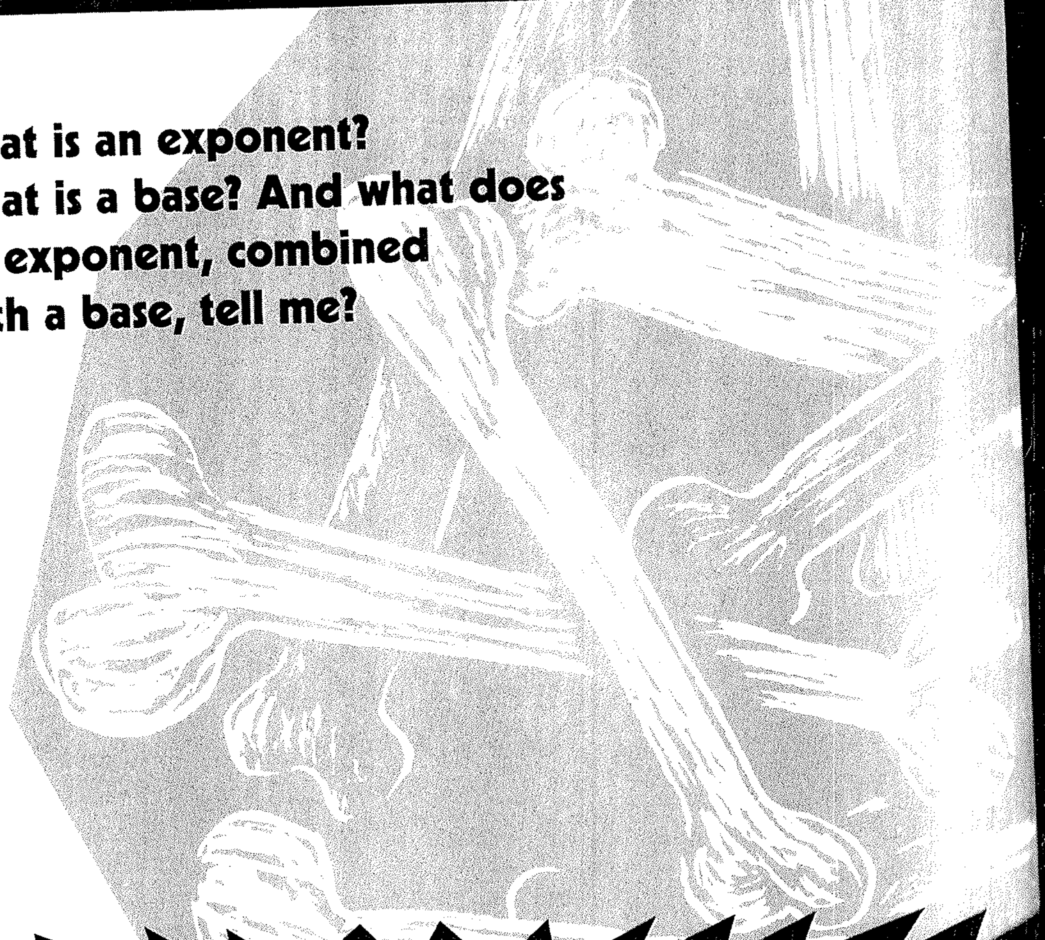
Q: What is an exponent? What is a base? And what does an exponent, combined with a base, tell me?



A: That little number that looks like it went through a shrinking machine is called an **exponent**. It sits on the right shoulder of a normal-sized number called the **base**. In short, **an exponent tells you how many times the base is multiplying itself.** The base along with its exponent is called an **exponential term.**

Definitions

a^x — a is the base, x is the exponent

5^3 — 5 is the base, 3 is the exponent

a^x means **a** is multiplying itself **x** times.

5^3 means **5** is multiplying itself **3** times.

Examples of exponential terms

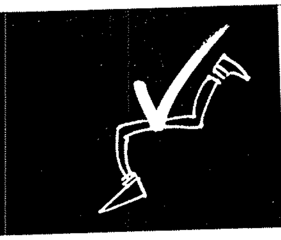
$a^2 = a \cdot a$

$a^5 = a \cdot a \cdot a \cdot a \cdot a$

$3^2 = 3 \cdot 3 = 9$

$3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 243$

$(ac)^5 = (ac) \cdot (ac) \cdot (ac) \cdot (ac) \cdot (ac)$



Try working out the value of these exponential terms:

a) 5^2
b) v^3
c) 10^3
d) $(mn)^4$
e) $(2x)^2$

Answers:
a) 25
b) $v \cdot v \cdot v$
c) 1, 000
d) $(mn) \cdot (mn) \cdot (mn) \cdot (mn)$
e) $(2x) \cdot (2x)$

Q: Why is an exponent sometimes called a "power"?

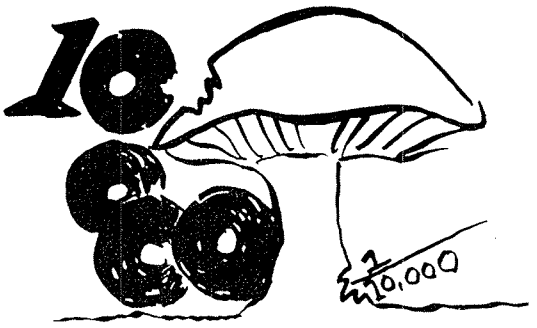
A: Exponents are called powers because they have amazing power to boost or shrink the value of numbers.

Remember the mushroom in *Alice in Wonderland*? When Alice took a bite out of one side of the mushroom, she became much larger; when she nibbled from the other side, she became much smaller. Exponents are a lot like the mushroom; used one way, they can make a number much larger — used another way, they can make a number much smaller.

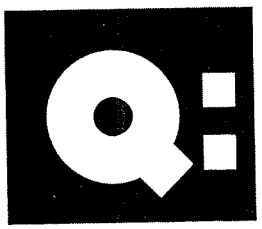
Some examples of the power of exponents

Take a number like **100**. When you raise **100** to the second power ($\mathbf{100^2}$), the value of this term suddenly shoots up to **10,000**.

On the other hand, when you raise **100** to the negative second power ($\mathbf{100^{-2}}$), its value suddenly shrinks to **1/10,000**, a teensy-tiny fraction.



In this section you'll learn why exponents have such tremendous power. You'll also learn some fundamental rules for working with exponents. First you'll learn the **same-base product rule** and the **same-base quotient rule**, as well as the meaning of $\mathbf{a^1}$ and of $\mathbf{a^0}$. After that you'll discover the strange meaning of a **negative exponent**. Finally you'll learn the **exponent-to-exponent rule**, the **product-to-exponent rule** and the **quotient-to-exponent rule**. All along your trek through the world of exponents, you'll learn how to perform **algebraic operations** with exponents, so that by the time you finish this section, you'll be an exceptional exponent expert.



What would be the value of something raised to the first power? In other words, what would be the value of

$$a^1$$

Anything raised to the first power is simply itself.

In other words, $a^1 = a$

From another viewpoint, a^1 can be thought of as **a** — one time. And in general, anything to the first power is just itself. In other words, you could say something ridiculous, like: **shoelace**^{1} = **shoelace**, and it would still be true! (See note to the right.)

More examples of terms to the first power

$3^1 = 3$

$(-5)^1 = -5$

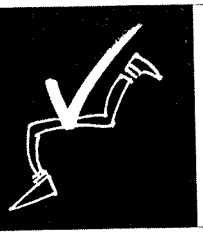
$(3/4)^1 = 3/4$

$(\sqrt{2})^1 = \sqrt{2}$

$(ac)^1 = ac$

$(7ac)^1 = 7ac$

From time to time, this guidebook will offer examples of algebraic rules that use goofy ideas, like crocodile, elephant, donut, etc. Textbooks don't use ideas like these, so why does this guidebook do so? Two reasons. First, many people can better survive the trek through the barren, abstract algebraic desert if they can visualize something, even if it's something ridiculous like an elephant being raised to the second power. Secondly, these examples help you see the general pattern for algebraic rules and principles. When you read goofy examples, bear in mind that the rule being discussed applies **to any algebraic term whatsoever!** In other words, if the rule works for something as ridiculous as an elephant, it will certainly work for anything you'll run across in algebra.



Simplify these terms:

a) x^1

b) 11^1

c) 1^1

d) $(-.032)^1$

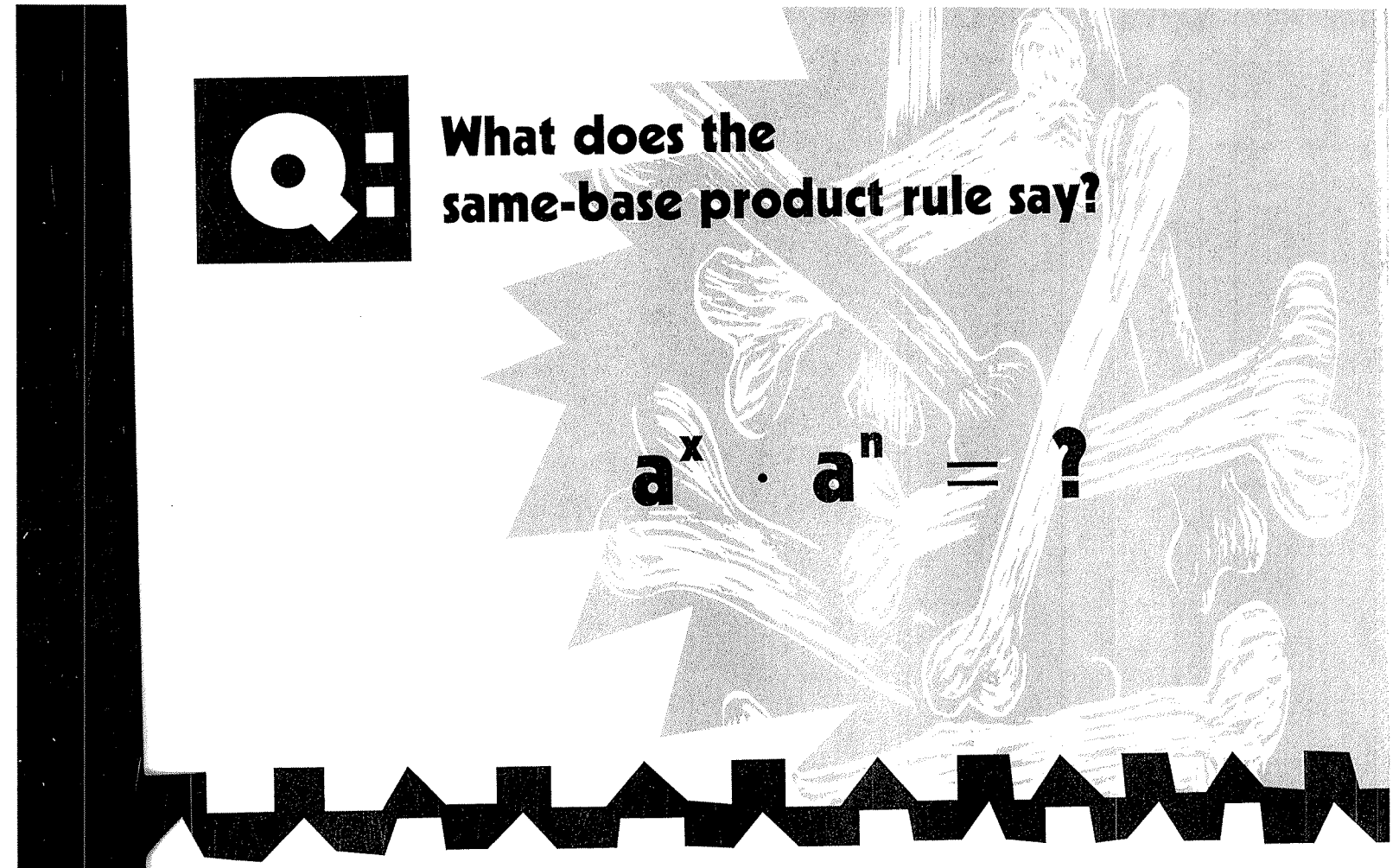
e) $(6x/y)^1$

Answers:

a) x d) $-.032$

b) 11 e) $6x/y$

c) 1



Q: What does the same-base product rule say?

$$a^x \cdot a^n = ?$$

A: The same-base product rule says this:

$$a^x \cdot a^n = a^{x+n}$$

Or in plain English: **when you're multiplying exponential terms whose bases are the same, keep that base and add the exponents.**

Examples of the same-base product rule

$$m^r \cdot m^u = m^{r+u}$$

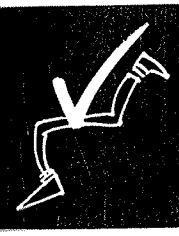
$$7^r \cdot 7^u = 7^{r+u}$$

$$m^5 \cdot m^3 = m^{5+3} = m^8$$

$$7^5 \cdot 7^3 = 7^{5+3} = 7^8$$

Goofy example

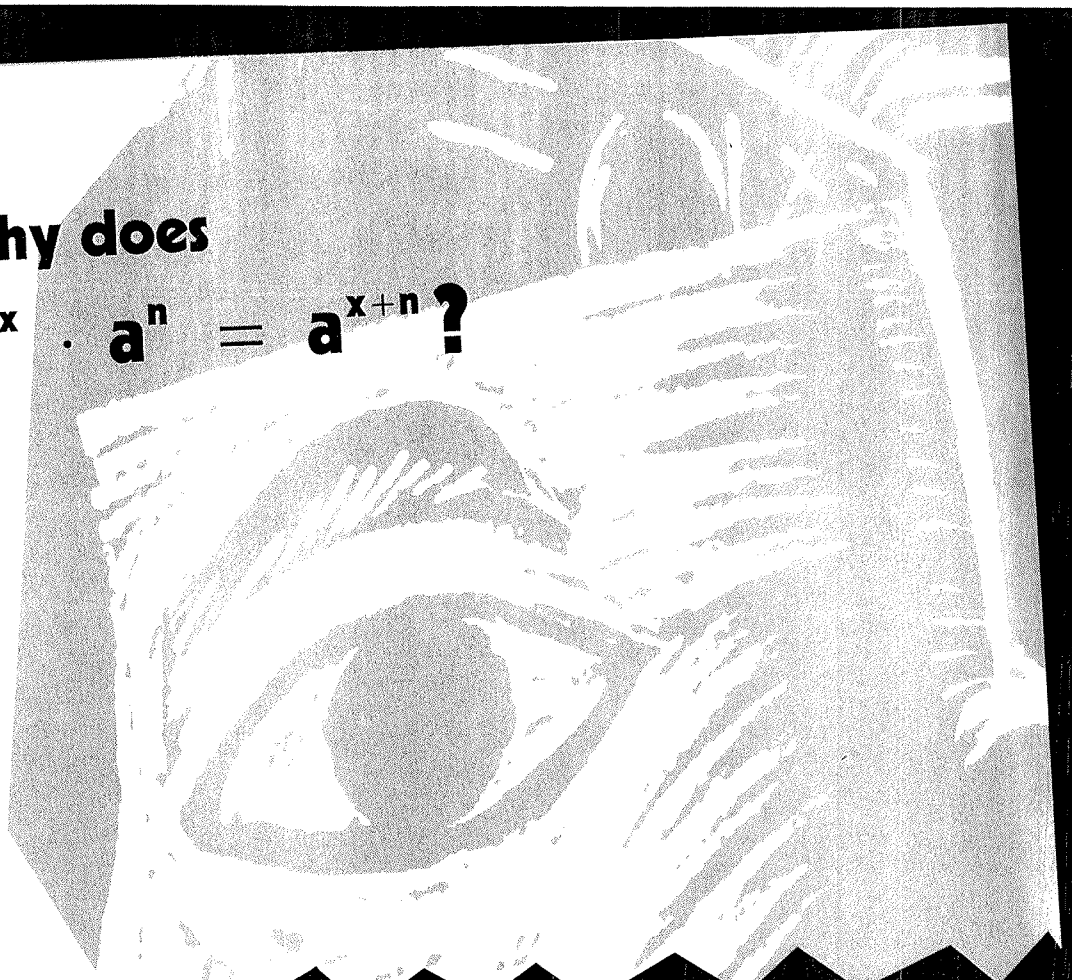
$$\text{turnip}^2 \cdot \text{turnip}^5 = \text{turnip}^{2+5} = \text{turnip}^7$$



Now try simplifying these expressions:

a) $x^3 \cdot x^4$
b) $3^5 \cdot 3^7$
c) $w^z \cdot w^y \cdot w^x$
d) $5^2 \cdot 5^3 \cdot 5^4$
e) $\text{muffin}^6 \cdot \text{muffin}^8$

Answers:
a) x^7 d) 5^9
b) 3^{12} e) muffin^{14}
c) w^{z+y+x}



Q: Why does $a^x \cdot a^n = a^{x+n}$?

A: Good question. It's always best to understand the "why" of things in math. So let's try to understand the same-base product rule by showing that:

$$a^3 \cdot a^2 = a^{3+2} = a^5$$

Let's begin with the expression: $a^3 \cdot a^2$. Now ... the very definition of an exponent (**p. 86**) tells you that:

$$a^3 \cdot a^2 = (a \cdot a \cdot a) \cdot (a \cdot a)$$

Since all terms are multiplying one another, you can remove parentheses: $(a \cdot a \cdot a) \cdot (a \cdot a) = a \cdot a \cdot a \cdot a \cdot a$

Again the definition of an exponent tells you that:

$$a \cdot a \cdot a \cdot a \cdot a = a^5$$

Since all these terms are equal to each other, you can use the transitive property (**p. 11**) to say that what you start with is equal to what you end up with, or: $a^3 \cdot a^2 = a^5$. This is what you wanted to show, so the job is done.

So what does this say? It tells you that any problem of the form $a^x \cdot a^n = a^{x+n}$ is basically an **addition** problem. In it, you're **adding** numbers of terms that are being multiplied.

In this example, **3 a**'s multiplied together times **2 a**'s multiplied together gives you **5 a**'s multiplied together, in the same way that $3 + 2 = 5$

Q: What does the same-base quotient rule say?

$$\frac{a^x}{a^n} = ?$$

A: The same-base quotient rule tells you this:

$$\frac{a^x}{a^n} = a^{x-n}$$

Or, put more simply: **when you're dividing exponential terms whose bases are the same, keep that base and subtract the exponents.**

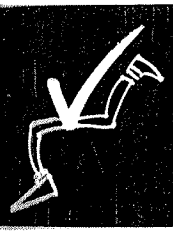
Examples of the same-base quotient rule

$$\frac{m^r}{m^u} = m^{r-u} \qquad \frac{m^5}{m^3} = m^{5-3} = m^2$$

$$\frac{7^r}{7^u} = 7^{r-u} \qquad \frac{7^5}{7^3} = 7^{5-3} = 7^2$$

Silly example

$$\frac{\text{kumquat}^9}{\text{kumquat}^4} = \text{kumquat}^{9-4} = \text{kumquat}^5$$



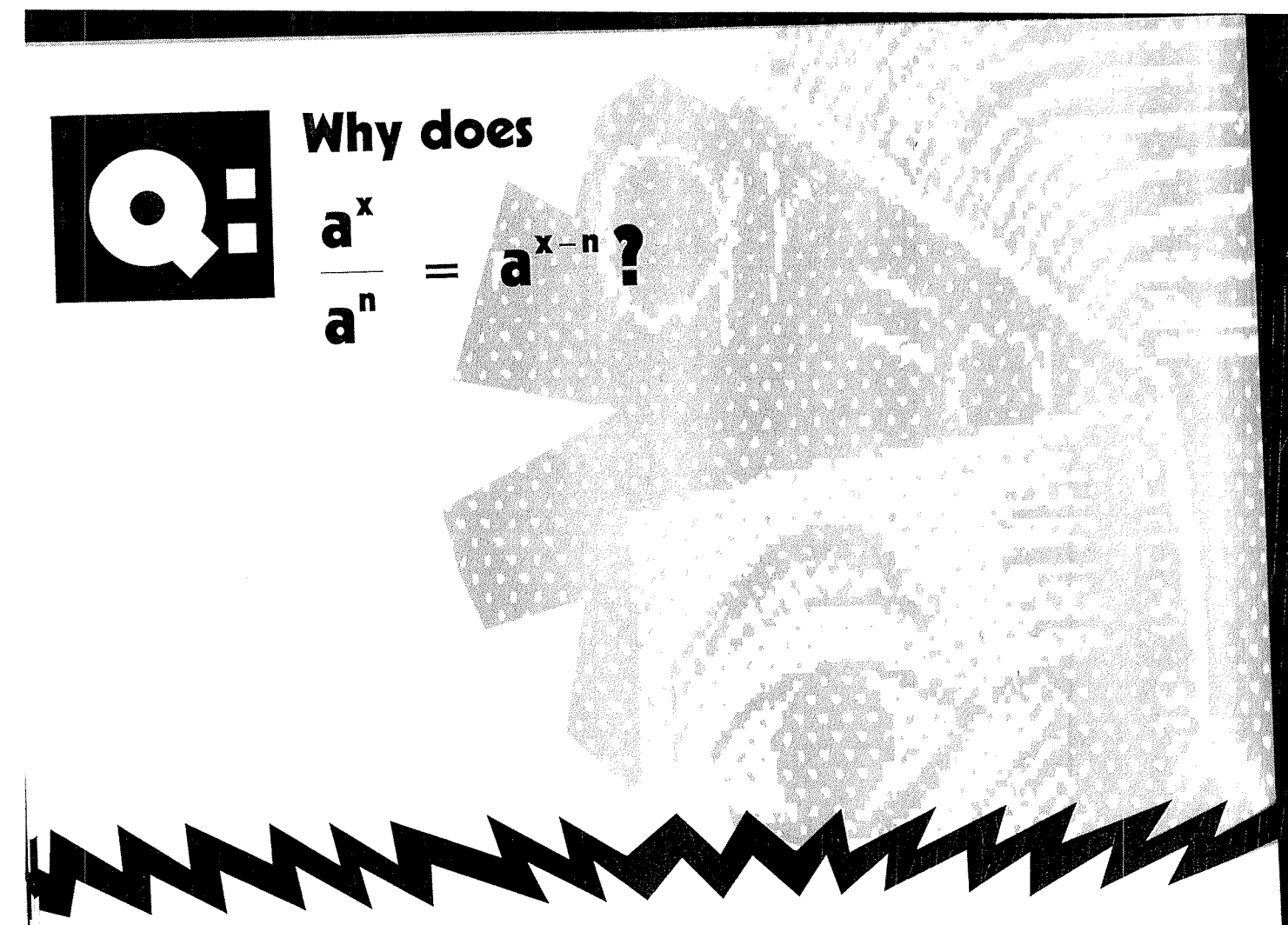
Now try to simplify these terms:

a) $\frac{3^{11}}{3^6}$ c) $\frac{p^{20}}{p^1}$

b) $\frac{w^r}{w^v}$ d) $\frac{\text{flea}^{13}}{\text{flea}^9}$

Answers:
a) 3^5 c) p^{19}
b) w^{r-v} d) flea^4

Q: Why does $\frac{a^x}{a^n} = a^{x-n}$?



A: Let's try to understand the same-base quotient rule by showing that:

$$\frac{a^5}{a^3} = a^{5-3} = a^2$$

Let's start out with: $\frac{a^5}{a^3}$

By the definition of an exponent (**p. 86**), you know that:

$$\frac{a^5}{a^3} = \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a \cdot a}$$

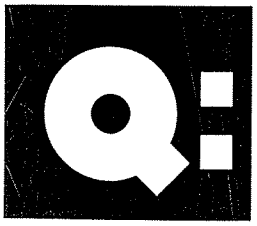
And from the rules of cancelling (**pp. 168-171**), you can cancel three **a**'s on the top of the fraction with three **a**'s on the bottom, to get:

$$\frac{\cancel{a} \cdot \cancel{a} \cdot \cancel{a} \cdot a \cdot a}{\cancel{a} \cdot \cancel{a} \cdot \cancel{a}} = \frac{a \cdot a}{1}$$

But $\frac{a \cdot a}{1} = a \cdot a$, and $a \cdot a = a^2$, by the definition of an exponent. So again you use the transitive property (**p. 11**) to show that what you start with is equal to what you wind up with. That is: $\frac{a^5}{a^3} = a^2$

In short, any problem like $\frac{a^x}{a^n}$ is basically a **subtraction** problem. In it, you're **subtracting** the number of terms in the denominator from the number of terms in the numerator.

In this example, **5 a**'s multiplied together divided by **3 a**'s multiplied together gives you **2 a**'s multiplied together, just as sure as **5 − 3 = 2**



Now I see that the same-base product and quotient rules work when the bases are the same. But don't they also work when the bases are different?

In other words, if I see something like $a^2 \cdot c^3$, I can just simplify it as: $(ac)^5$

And if I see $\frac{a^5}{c^3}$, I can simplify it as: $\left(\frac{a}{c}\right)^2$

Uh ... can't I?

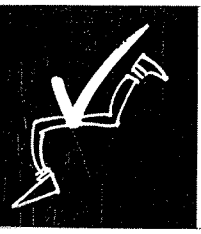
No, no. Wrong, wrong! You can use these rules only when the bases are **exactly, completely and 100% the same**. That's why the phrase **same-base** is part of their names. So please — **don't even dream about using these rules if the bases are not entirely the same**. Below you'll see this idea fleshed out.

Same-base product rule

A term like $a^2 \cdot a^3$ can be simplified as a^5 only because both terms being multiplied have exactly the same base, **a**. But a term like $a^2 \cdot c^3$ cannot be simplified because the two terms have different bases: the first has a base of **a**; the second a base of **c**. So this expression must be left as is.

Same-base quotient rule

A term like $\frac{a^5}{a^3}$ can be simplified as a^2 because both bases are **a**'s. But a term like $\frac{a^5}{c^3}$ cannot be simplified because the terms have different bases: the numerator has a base of **a**; the denominator, a base of **c**. So this expression also must be left as is.



Tell whether or not these terms can be simplified. If they can, go ahead and simplify them.

a) $3^x \cdot x^3$ c) $\frac{b^y}{7^y}$ e) $\frac{m^5}{m^4}$

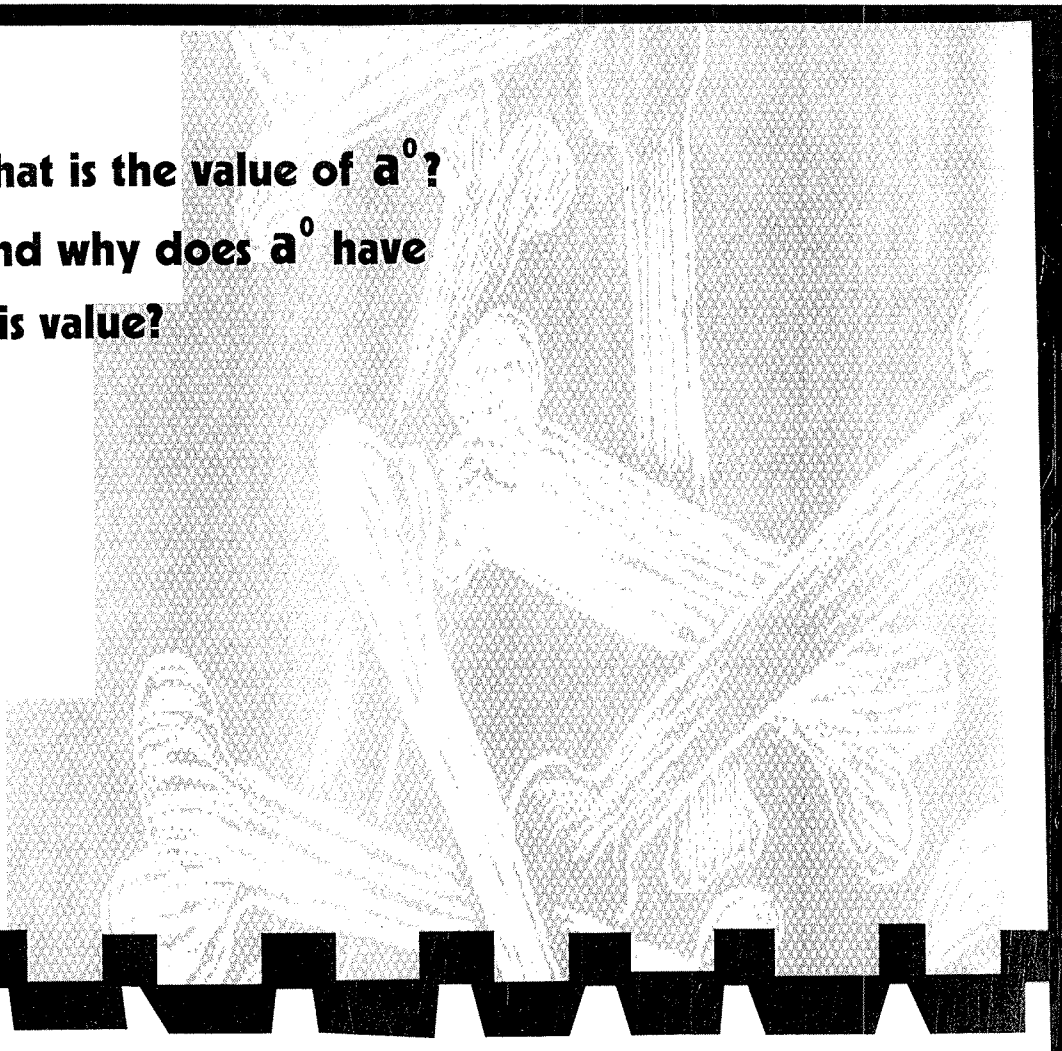
b) $a^4 \cdot a^4$ d) $r^r \cdot r^r$ f) $(2m)^7 \cdot (2n)^8$

Answers:

a) can't d) can, r^{r+r}

b) can, a^8 e) can, m^1

c) can't f) can't



Q: What is the value of a^0? And why does a^0 have this value?

A: Any term raised to the zero power equals **1**, not **0**, as you might expect. Or, to put this in math-talk: no matter what value **a** might have,

$$a^0 = 1$$

Examples

$$3^0 = 1$$

$$(.6)^0 = 1$$

$$(-4)^0 = 1$$

$$(x/y)^0 = 1$$

$$\pi^0 = 1$$

Why do terms raised to the zero power equal **1**? Let's try to understand this idea by examining a little fraction: a^5/a^5

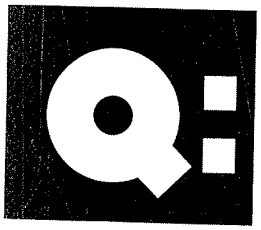
If you have faith in the same-base quotient rule (**p. 91**), you'd probably have to agree that: $a^5/a^5 = a^{5-5}$, and by simple subtraction, you know that: $a^{5-5} = a^0$. Then, using the transitive property (**p. 11**), you can see that: $a^5/a^5 = a^0$

But why not look at a^5/a^5 in a different way — as a fraction whose numerator and denominator are the same. From elementary math you know that whenever a fraction's numerator and denominator are the same, that fraction equals **1**. So that means: $a^5/a^5 = 1$

Now look at your two findings. First you saw that: $a^5/a^5 = a^0$ And then you saw that: $a^5/a^5 = 1$. And if you just use the transitive property one more time, it tells you that: $a^0 = 1$

Since **a** stands for any number or term, this tells you that any term raised to the zero power always equals **1**.

Note: there's one exception to this rule. $0^0 \neq 1$. Instead, 0^0 is "undefined" because it involves dividing by zero, which as you saw earlier (**p. 30**) was an algebraic "no-no."



When I have a term raised to the zero power showing up in my answer, I may as well ignore it because it will always disappear from the final answer. Right?

No, not always. Terms to the zero power do disappear from final answers when they are multiplying or dividing other terms. This is because any term to the zero power equals **1**, and multiplying or dividing any term by **1** just gives you back the term you started with. But when terms to the zero power are being added to or subtracted from other terms, they do not disappear; they affect the final answer. Below are some illustrations of this idea.

zero power term disappears from final answer

multiplication

$$\begin{aligned} & x^2c^3a^0 \\ = \; & x^2c^3 \cdot 1 \\ = \; & x^2c^3 \end{aligned}$$

division

$$\begin{aligned} & \frac{x^2}{a^0} \\ = \; & \frac{x^2}{1} \\ = \; & x^2 \end{aligned}$$

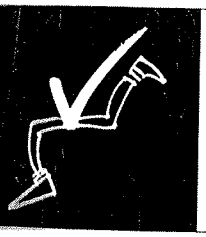
zero power term affects final answer

addition

$$\begin{aligned} & 4 + x^0 \\ = \; & 4 + 1 \\ = \; & 5 \end{aligned}$$

subtraction

$$\begin{aligned} & 4 - x^0 \\ = \; & 4 - 1 \\ = \; & 3 \end{aligned}$$



Tell what the final answer would be:

a) $x^6y^0z^9$

b) $x^6 + x^0$

c) $\dfrac{a^2b^4c^0}{m^4n^3}$

d) $8 - z^0$

e) $v^0 + e^0 + w^0$

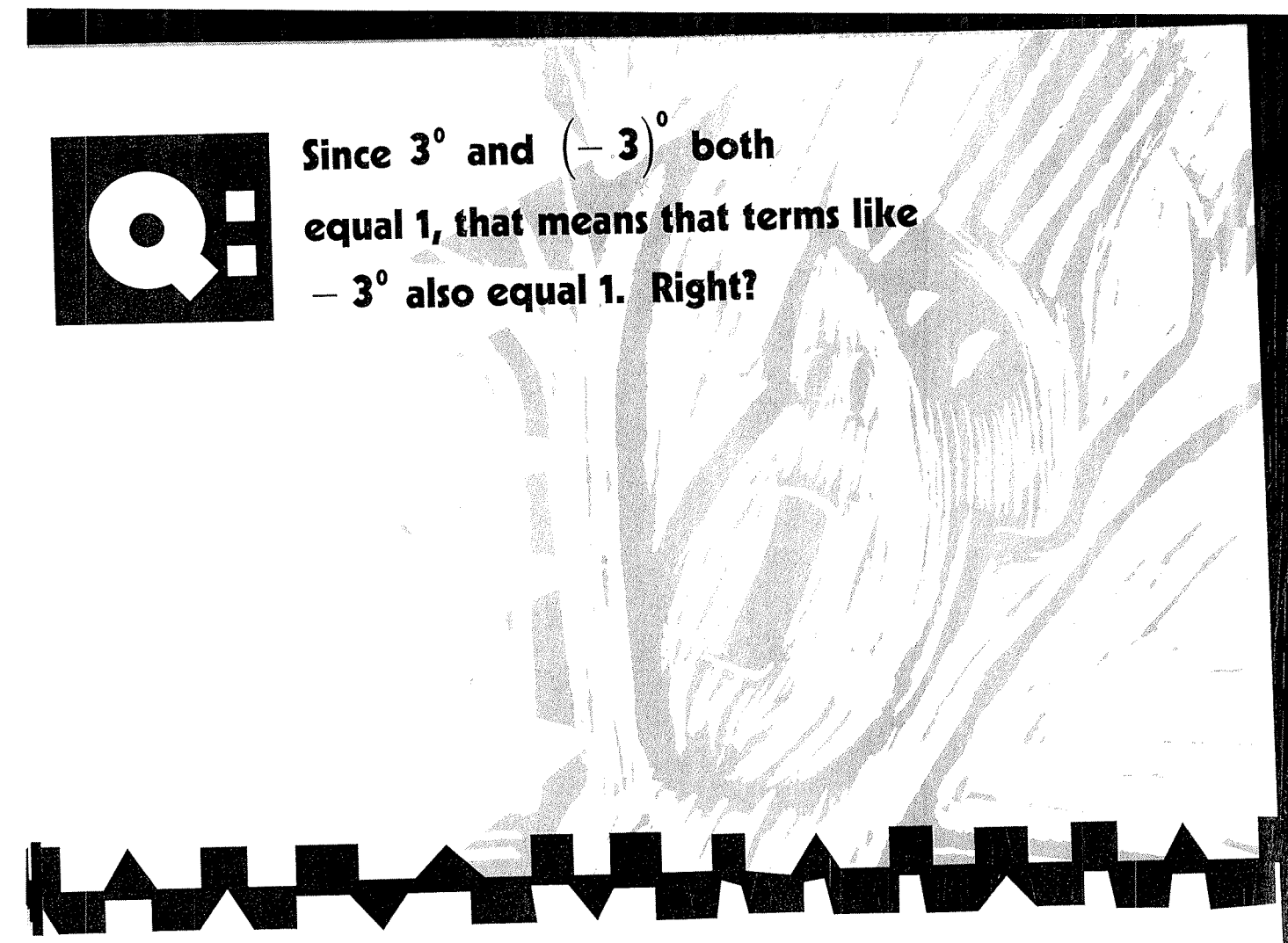
f) $4a^0b^0c^0d^0$

Answers:

a) x^6z^9 d) 7

b) $x^6 + 1$ e) 3

c) $\dfrac{a^2b^4}{m^4n^3}$ f) 4



Q: Since 3^0 and $(-3)^0$ both equal 1, that means that terms like -3^0 also equal 1. Right?

A: Sorry, but no. But if you just think back to the order of operations, you'll quickly see why this is not the case.

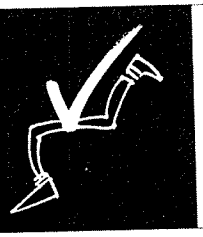
With both 3^0 and $(-3)^0$ you have only one operation to work out, and that is to raise the term to the exponent. Since any term raised to the zero power equals **1**, both of these terms equal **1**.

But when you look at -3^0, you really should think of it as: $-(3^0)$. When you view the term this way, you can see that you have two operations to do:

1st) working out the **exponent**, and

2nd) multiplying by -1. Since **E** comes before **M**, you first work out the **exponent**, then **multiply** by -1, as shown to the right:

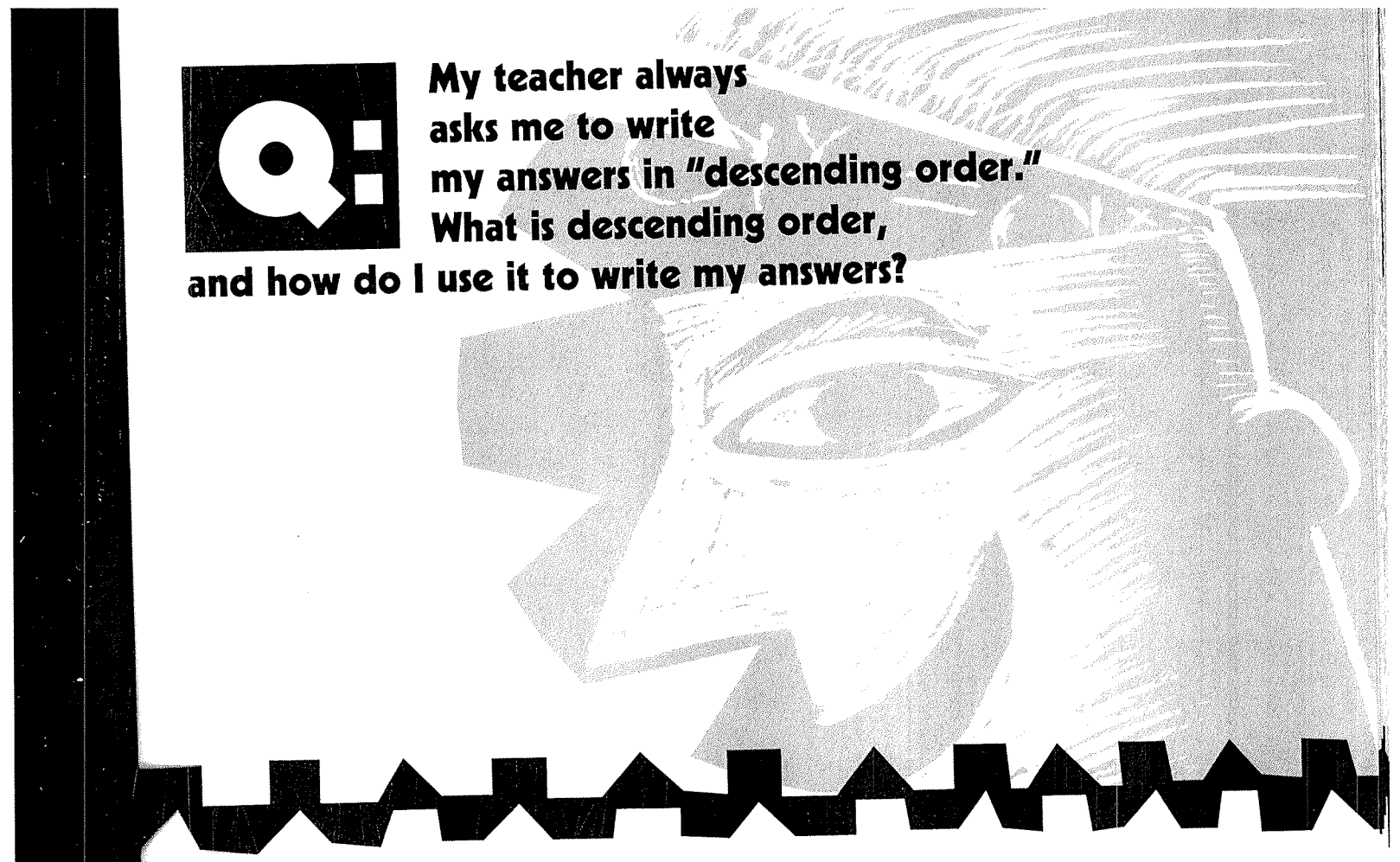
$$-(3^0) = -(1) = -1$$



Try stating the value of these terms:

a) -4^0
b) $(-4)^0$
c) 4^0
d) $7+4^0$
e) $7-4^0$

Answers: a) -1 b) 1 c) 1 d) 8 e) 6



Q: My teacher always asks me to write my answers in "descending order." What is descending order, and how do I use it to write my answers?

A: To understand descending order, think about what it means to walk down, or to descend, a short flight of stairs.

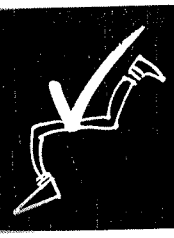
You start out, say, three steps above the floor, and you descend the stairs one at a time till you reach the bottom. Now imagine that instead of real stairs, you're walking down a stairway of exponents. You start out at the highest exponent, then drop to the next highest one, and so on, until finally you reach the smallest exponent. When you arrange a bunch of terms this way, making the exponents descend across the page from left to right, you're arranging them in descending order.

Below you'll see the difference between scrambled terms and terms in descending order.

Terms scrambled		Same terms in descending order
$-2a^2 + 8a^5 - 7 - 4a^3 + 3a + 6a^4$	$\rightarrow$	$+ 8a^5 + 6a^4 - 4a^3 - 2a^2 + 3a - 7$
$-3m + 6m^4 - 9 + 8m^7$	$\rightarrow$	$+ 8m^7 + 6m^4 - 3m - 9$

Note: — A variable with no exponent shown has an invisible exponent of **1**. (ex.: **a** means a^1, **m** means m^1)

— Number terms always go last.



Try arranging these terms in descending order:

a) $-5 + 3x$

b) $+4y - y^2 - 11$

c) $17 - 3z^4 + 12z - 2z^3$

d) $-n^3 + 6n^5 - 2 + 3n^2 + 8n^4$

Answers:

a) $+3x - 5$

b) $-y^2 + 4y - 11$

c) $-3z^4 - 2z^3 + 12z + 17$

d) $+6n^5 + 8n^4 - n^3 + 3n^2 - 2$