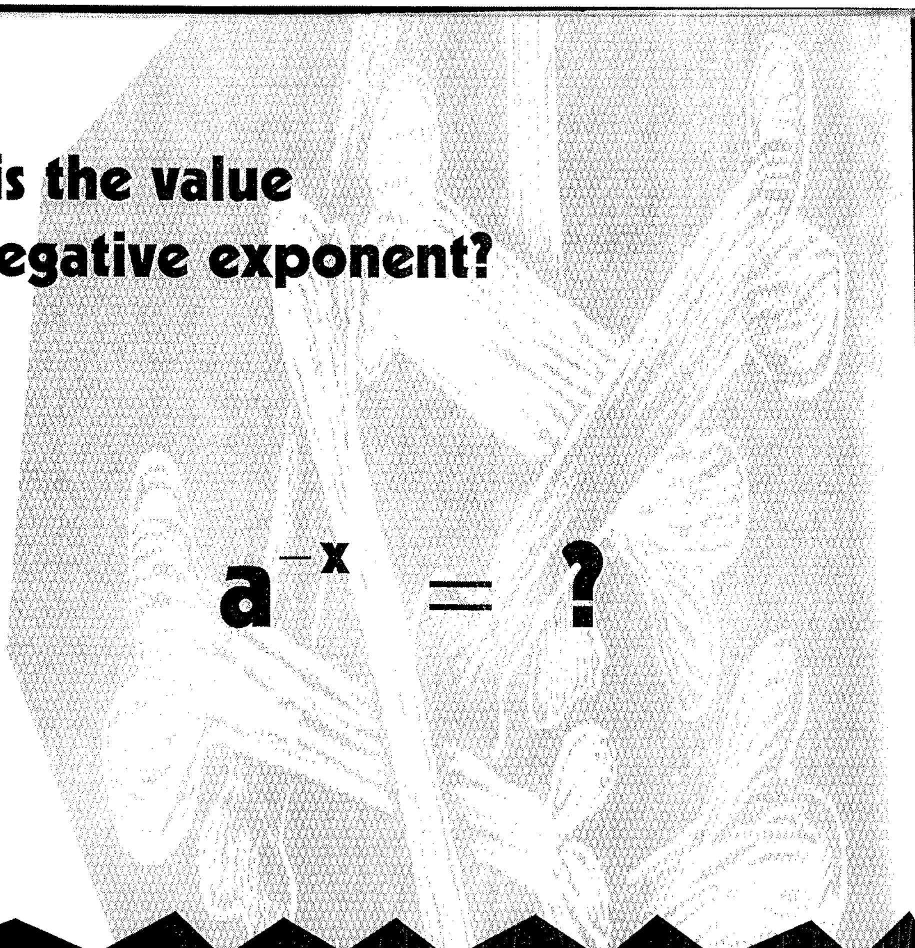
Q: What is the value of a negative exponent?



$$a^{-x} = ?$$

A: The value of a negative exponent is this:

$$a^{-x} = \frac{1}{a^x}$$

Or in simple English: whenever you have a term raised to a negative exponent, you can restate the value of that term by:

a) pushing it across the fraction bar, and

b) making its exponent positive.

Examples of negative exponents

$$m^{-r} = \frac{1}{m^r} \qquad 6^{-r} = \frac{1}{6^r}$$

$$m^{-3} = \frac{1}{m^3} \qquad 6^{-3} = \frac{1}{6^3} = \frac{1}{216}$$

Silly example

$$frog^{-4} = \frac{1}{frog^4}$$

Try simplifying these terms:

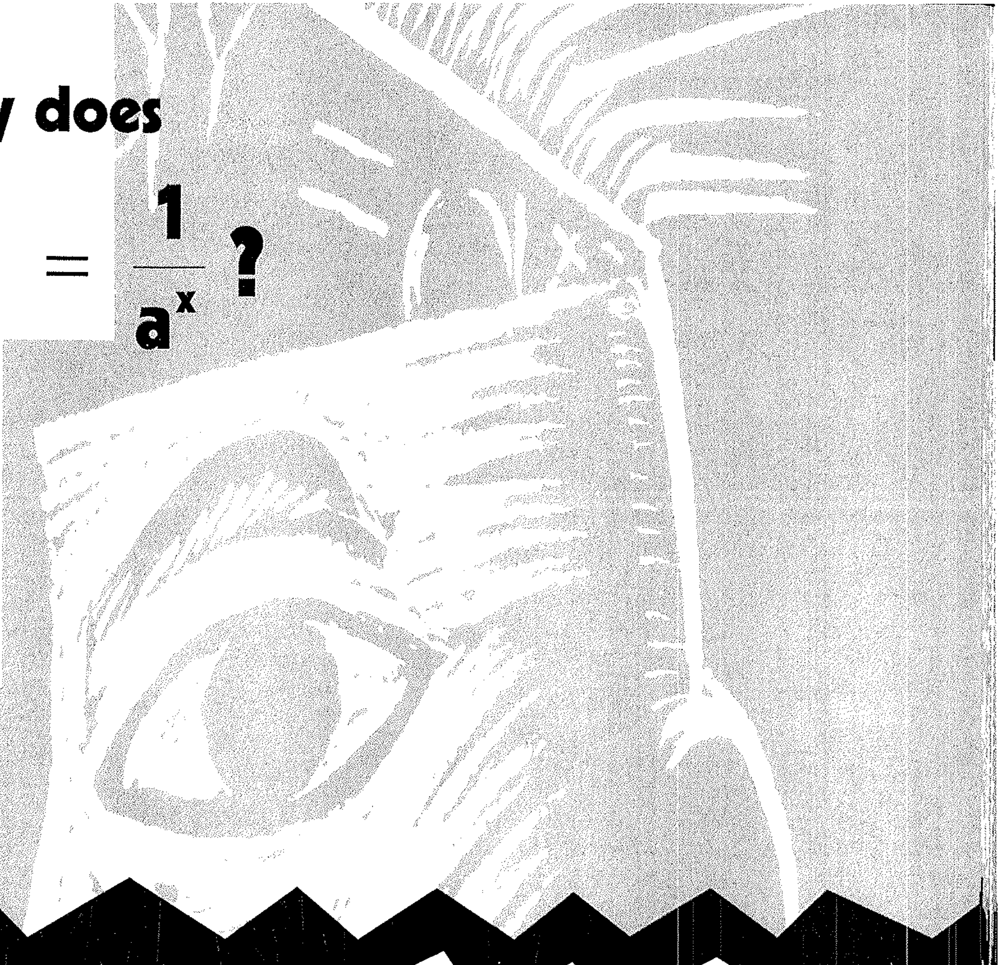
a) v^{-5} **c)** $fish^{-7}$

b) x^{-a} **d)** 3^{-2}

Answers:

a) $\frac{1}{v^5}$ c) $\frac{1}{fish^7}$

b) $\frac{1}{x^a}$ d) $\frac{1}{3^2} = \frac{1}{9}$



Q: Why does $a^{-x} = \frac{1}{a^x}$?

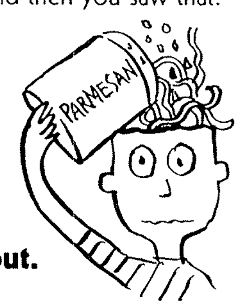
A: Let's try to understand this rule by showing that: $a^{-3} = \frac{1}{a^3}$

Let's start out with the term: a^2/a^5. Once again, if you have faith in the same-base quotient rule (**p. 91**), it would be hard to deny that: $a^2/a^5 = a^{2-5}$. And using simple subtraction, $a^{2-5} = a^{-3}$. So using the transitive property (**p. 11**), you can see that: $a^2/a^5 = a^{-3}$

But now let's look at a^2/a^5 in a different way, as $a \cdot a/a \cdot a \cdot a \cdot a \cdot a$. Using the rules for cancelling (**pp. 168-171**), you cancel the two **a**'s on the top with two of the **a**'s on the bottom, to give you: $\cancel{a} \cdot \cancel{a}/\cancel{a} \cdot \cancel{a} \cdot a \cdot a \cdot a = 1/a \cdot a \cdot a$. And using the definition of an exponent (**p. 86**), you can say that: $1/a \cdot a \cdot a = 1/a^3$. Since all these terms are equal to each other, you can use the transitive property again to say that: $a^2/a^5 = 1/a^3$

Now just look at the two main findings here. First you saw that: $a^2/a^5 = a^{-3}$. And then you saw that: $a^2/a^5 = 1/a^3$. Since all these terms are equal to each other, you can use the transitive property once more to say that: $a^{-3} = \frac{1}{a^3}$. And in general, any term to a negative exponent is the same as **1** over that term to a positive exponent.

By the way, if your brains feel like they're turning into spaghetti, it just means you're making a courageous effort to figure this stuff out.



Q: What is the value of $\frac{1}{a^{-x}}$?

A: The term's value is this: $\frac{1}{a^{-x}} = a^x$

Examples

$$\frac{1}{m^{-r}} = m^r \qquad \frac{1}{6^{-r}} = 6^r$$

$$\frac{1}{m^{-3}} = m^3 \qquad \frac{1}{6^{-3}} = 6^3 = 216$$

Silly example

$$\frac{1}{hen^{-5}} = hen^5$$

In other words: **1** over any term raised to a negative exponent is the same as that term raised to a positive exponent.

The reason this is true is quite simple. As we saw on the last page, $a^{-x} = \frac{1}{a^x}$. If we just substitute $\frac{1}{a^x}$ for a^{-x}, you see that: $\frac{1}{a^{-x}} = \frac{1}{\frac{1}{a^x}}$. And from basic arithmetic, when you divide a term by a fraction, you just multiply the term by the flip (or, reciprocal) of the fraction. So...

$$\frac{1}{a^{-x}} = \frac{1}{\frac{1}{a^x}} = 1 \cdot \frac{a^x}{1} = a^x$$

Try simplifying these terms:

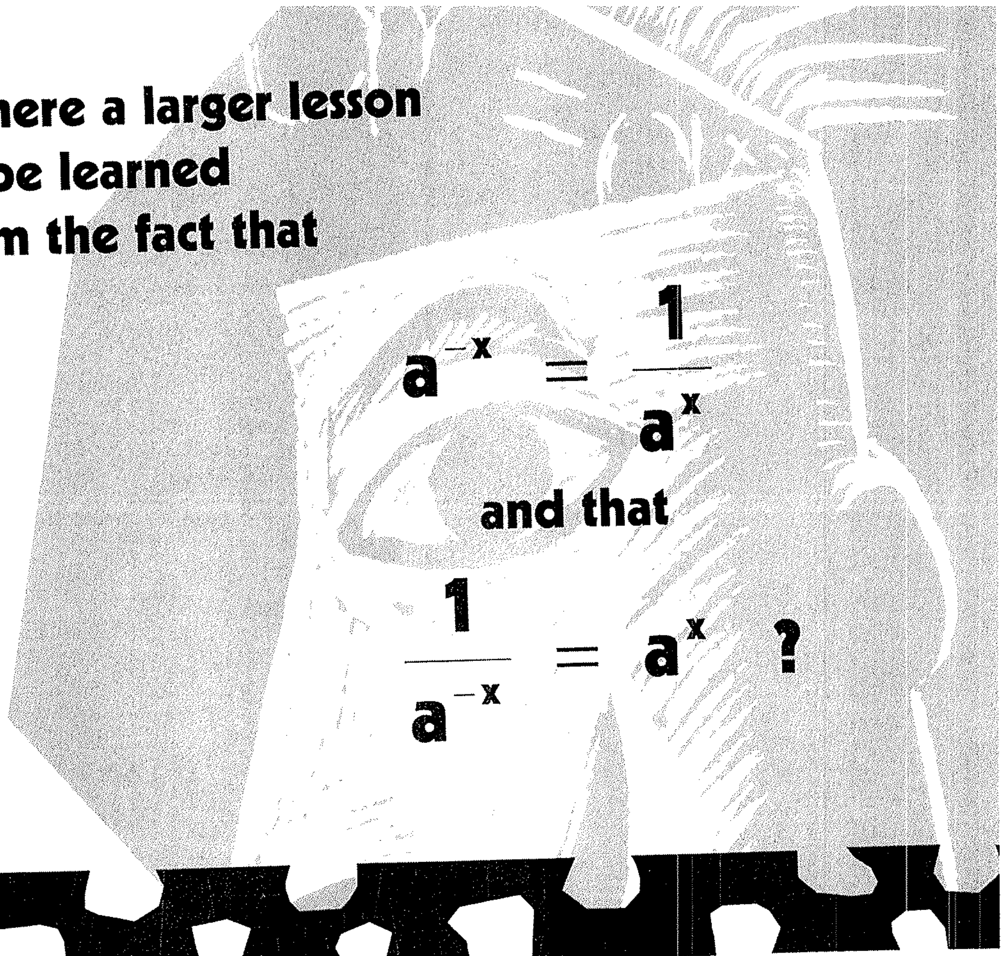
a) $\frac{1}{w^{-3}}$ c) $\frac{1}{2^{-3}}$

b) $\frac{1}{tadpole^{-7}}$ d) $\frac{1}{c^{-d}}$

Answers:

a) w^3 c) $2^3 = 8$

b) $tadpole^7$ d) c^d



Q: Is there a larger lesson to be learned from the fact that

$$a^{-x} = \frac{1}{a^{x}}$$

and that

$$\frac{1}{a^{-x}} = a^{x} \quad ?$$

A: Actually there is a big lesson, and it is this: Whenever you push an exponential term across the fraction bar, the exponent's sign changes. If the exponent's sign was positive, it becomes negative; if its sign was negative, it becomes positive. Below are illustrations of this idea.

exponent's sign changes from negative to positive

$$\frac{5a^{-3}}{7} = \frac{5}{7a^{3}}$$

$$\frac{1}{2c^{-4}} = \frac{c^{4}}{2}$$

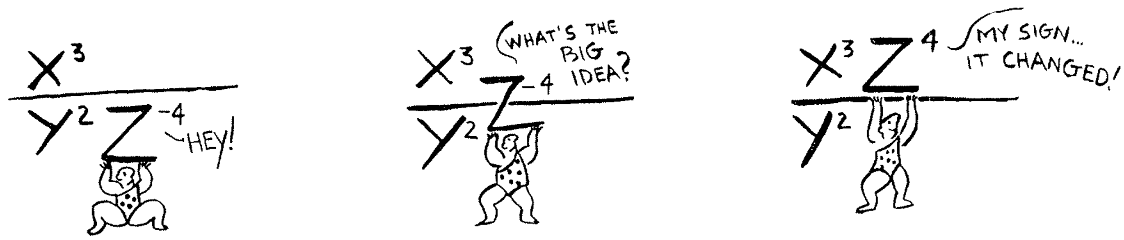
$$\frac{\text{popsicle}^{-2}}{\text{lollipop}^{-5}} = \frac{\text{lollipop}^{5}}{\text{popsicle}^{2}}$$

exponent's sign changes from positive to negative

$$\frac{8x^{4}}{11} = \frac{8}{11x^{-4}}$$

$$\frac{1}{6w^{3}} = \frac{w^{-3}}{6}$$

$$\frac{\text{popsicle}^{8}}{\text{lollipop}^{3}} = \frac{\text{lollipop}^{-3}}{\text{popsicle}^{-8}}$$

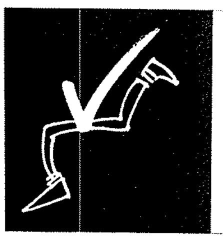


Q: My teacher asks me to write my answers without negative exponents. Can I use the idea from the previous page to make exponents positive in my answer?

A: Yes, just move your terms so that the exponents become positive.
Follow the two steps below.

Make the exponents positive: **A)** $\frac{4a^{-2}}{c^3}$, **B)** $\frac{4a^2}{c^{-3}}$, **C)** $\frac{4a^{-2}}{c^{-3}}$

Steps	Example A	Example B	Example C
1st) Original problem.	$\frac{4a^{-2}}{c^3}$	$\frac{4a^2}{c^{-3}}$	$\frac{4a^{-2}}{c^{-3}}$
2nd) Move the term with the negative exponent across the fraction bar and make its sign positive.	$=\frac{4}{a^2c^3}$	$=4a^2c^3$	$=\frac{4c^3}{a^2}$



Try rewriting these terms so that they have only positive exponents:

a) $4a^{-2}b^{-3}$
b) $8^{-2}x^{-3}y^{-4}$
c) $\frac{6}{a^{-2}m^{-4}}$
d) $\frac{x^{-3}}{a^{-4}}$
e) $\frac{a^{-1}b^{-2}c^{-3}}{d^{-3}e^{-2}f^{-1}}$

Answers:
a) $\frac{4}{a^2b^3}$ b) $\frac{1}{8^2x^3y^4}$ c) $6a^2m^4$ d) $\frac{a^4}{x^3}$ e) $\frac{d^3e^2f}{ab^2c^3}$

Q: Now I see how to get rid of negative exponents. But I also need to know what to do when I have more than one term in the numerator or in the denominator. In other words, how would I simplify an expression like this:

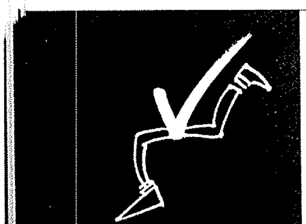
$$\frac{x^{-4}x^{10}x^{0}}{x^{13}x^{-2}}$$

A: Just follow the three steps illustrated to the right.

Simplify:

$$\frac{x^{-4}x^{10}x^{0}}{x^{13}x^{-2}}$$

Steps	Example
1st) Use the same-base product rule (**p. 89**) on the terms in the numerator and on the terms in the denominator.	$\frac{x^{-4}x^{10}x^{0}}{x^{13}x^{-2}} = \frac{x^{-4+10+0}}{x^{13+-2}} = \frac{x^{6}}{x^{11}}$
2nd) Move the term with the smaller exponent.	$= \frac{1}{x^{-6} \cdot x^{11}}$
3rd) Use the same-base product rule again.	$= \frac{1}{x^{5}}$



Now try simplifying these terms in the same way:

a) $\frac{a^{-5}a^{9}}{a^{11}a^{0}a^{-2}}$

b) $\frac{4^{10} \cdot 4^{-6}}{4^{-20} \cdot 4^{5}}$

c) $\frac{n^{2}n^{3}n^{4}}{n^{-4}n^{-3}n^{-2}}$

d) $\frac{r^{18}r^{18}}{r^{-2}r^{-2}}$

Answers: a) $\frac{1}{a^{5}}$ b) 4^{19} c) n^{18} d) r^{40}

your input welcome!

Q: O.K., that wasn't too hard. But I can think of an even tougher problem — one in which altogether different bases stand on opposite sides of the fraction bar. How would I simplify a scary-looking thing like this:

$$\frac{a^{-4}c^{3}a^{2}c^{-12}}{c^{7}a^{0}c^{-5}a^{-10}}$$

A: Don't get too scared, because this is not so hard as it looks. Just follow the four steps to the right.

Simplify:

$$\frac{a^{-4}c^{3}a^{2}c^{-12}}{c^{7}a^{0}c^{-5}a^{-10}}$$

Steps	Example
1st) Group terms with the same bases.	$\frac{a^{-4}c^{3}a^{2}c^{-12}}{c^{7}a^{0}c^{-5}a^{-10}} = \frac{a^{-4}a^{2}}{a^{0}a^{-10}} \cdot \frac{c^{3}c^{-12}}{c^{7}c^{-5}}$
2nd) Use same-base product rule (**p. 89**).	$= \frac{a^{-2}}{a^{-10}} \cdot \frac{c^{-9}}{c^{2}}$
3rd) Move terms with the smaller exponents.	$= \frac{a^{-2}a^{10}}{c^{9}c^{2}}$
4th) Use same-base product rule.	$= \frac{a^{8}}{c^{11}}$

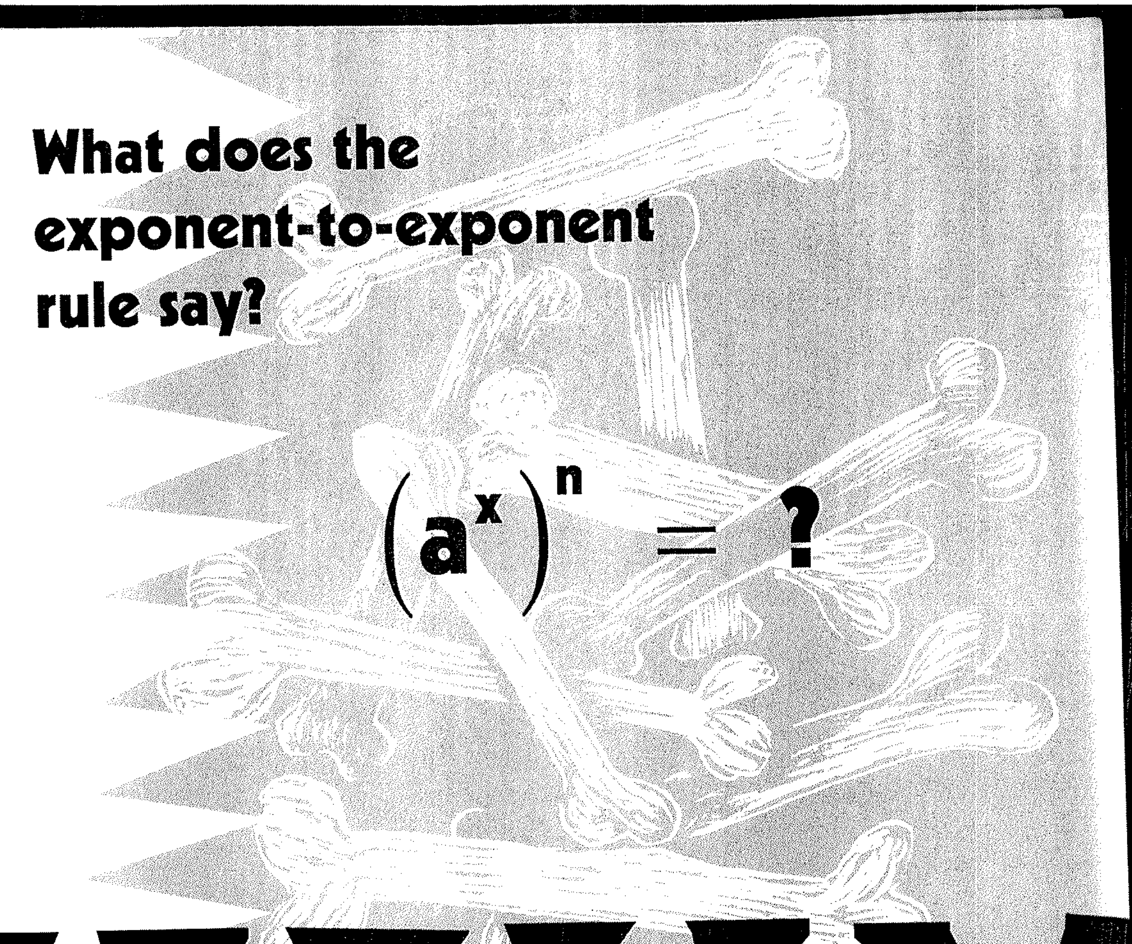
Now give these a shot:

a) $\frac{x^{3}w^{-4}}{w^{-8}x^{7}}$ c) $\frac{a^{0}n^{2}n^{-6}a^{-14}}{n^{-3}a^{4}a^{-5}n^{-9}}$

b) $\frac{u^{-4}v^{6}}{v^{-14}u^{3}v^{17}}$ d) $\frac{x^{-3}x^{2}r^{-10}x}{x^{8}r^{-2}r^{-3}r^{9}}$

Answers: a) $\frac{w^{4}}{x^{4}}$ b) $\frac{v^{3}}{u^{7}}$ c) $\frac{n^{8}}{a^{13}}$ d) $\frac{1}{x^{8}r^{14}}$

Q: What does the exponent-to-exponent rule say?



A: The exponent-to-exponent rule says this:

$$(a^x)^n = a^{x \cdot n}$$

Or: **when an exponential term itself gets raised to an exponent, keep the base and multiply the exponents.**

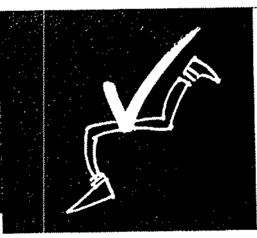
Examples of the exponent-to-exponent rule

$$(m^r)^u = m^{r \cdot u} \qquad (m^3)^{-5} = m^{3 \cdot -5} = m^{-15}$$

$$(5^r)^u = 5^{r \cdot u} \qquad (5^3)^5 = 5^{3 \cdot 5} = 5^{15}$$

Silly example

$$(\text{porcupine}^3)^4 = \text{porcupine}^{3 \cdot 4} = \text{porcupine}^{12}$$



Now try simplifying these using the exponent-to-exponent rule:

a) $(2^3)^5$ d) $(\text{gumball}^2)^5$

b) $(x^y)^u$ e) $(3^3)^{-3}$

c) $(a^{-5})^4$

Answers: a) 2^{15} b) $x^{y \cdot u}$ c) a^{-20} d) gumball^{10} e) 3^{-9}

Q: Now I know how to get rid of negative exponents in simple problems. But how do I get rid of negative exponents when the same variable stands on both sides of the fraction bar? In other words, how would I simplify problems like these:

$$\frac{c^3}{c^5}, \quad \frac{a^5}{a^{-3}} \quad \text{or} \quad \frac{n^{-3}}{n^{-5}}$$

A: It's easy if you remember this simple rule-of-thumb. **When it comes time to move exponential terms across the fraction bar, always move the term that has the smaller exponent.** Follow the steps to the right.

Steps	Example A	Example B	Example C
1st) For each term, notice which of the exponents is smaller.	$\frac{c^3}{c^5}\ (3 < 5)$	$\frac{a^5}{a^{-3}}\ (-3 < 5)$	$\frac{n^{-3}}{n^{-5}}\ (-5 < -3)$
2nd) Move the term with the smaller exponent across the fraction bar and change its sign.	$= \frac{1}{c^5 c^{-3}}$	$= \frac{a^5 \cdot a^3}{1}$	$= \frac{n^{-3} \cdot n^5}{1}$
3rd) Simplify by using the same-base product rule (**p. 89**).	$= \frac{1}{c^{5+-3}} = \frac{1}{c^2}$	$= a^{5+3}$ $= a^8$	$= n^{-3+5}$ $= n^2$

Try these:

a) $\frac{v^3}{v^{11}}$ c) $\frac{m^{-1}}{m^{-3}}$

b) $\frac{y^{-5}}{y^8}$ d) $\frac{5^{-13}}{5^{-20}}$

Answers:

a) $\frac{1}{v^8}$ b) $\frac{1}{y^{13}}$

c) m^2 d) 5^7