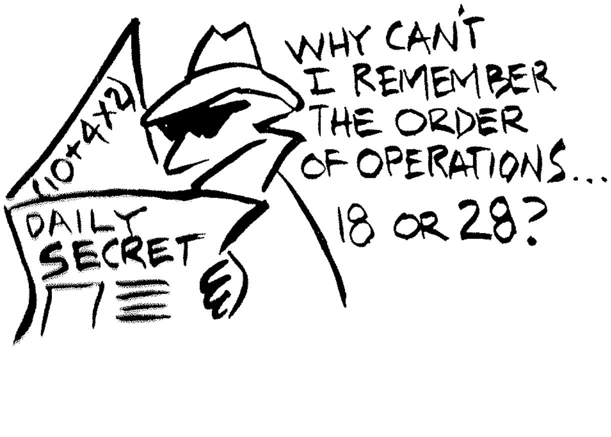
Q: What is the order of operations, and why do I need to learn it?

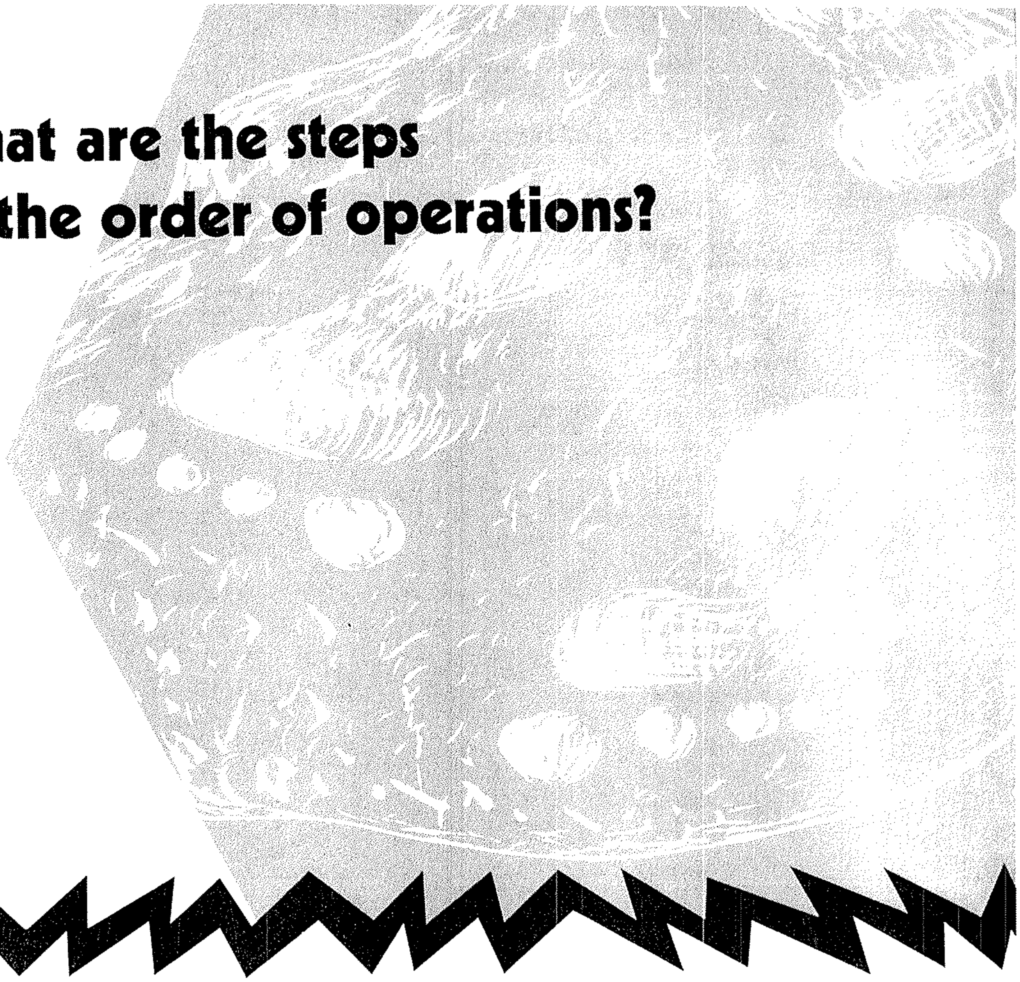
A: The order of operations is the order that tells you what to do first, then second, then third, etc. when you're simplifying an algebraic expression. In other words, if you ever find yourself scratching your head, wondering what you're supposed to do next — multiplication? subtraction? exponents? parentheses? — go back to the trusty old order of operations and you'll find your answer.



Why do you need this order of operations? Think of it this way. If you were a member of an organization that uses codes and secret messages, you'd need to be sure that if you sent someone a message, she or he could decode it properly. It would be pretty awful to send someone the message: "Meet me at 10 Oak Street at 4 p.m. with 2 of your buddies," only to have it read as "Meet me at 4 Oak Street at 2 p.m. with 10 of your buddies." Mathematical expressions are a little like secret messages. When someone in math writes out a statement like: $\mathbf{10 + 4 \cdot 2}$, s/he needs to make sure that this is decoded accurately. A person who doesn't know the order of operations wouldn't know whether to read $\mathbf{10 + 4 \cdot 2}$ as $\mathbf{14 \cdot 2}$ or as $\mathbf{10 + 8}$. Only one of these choices is correct. Do you know which one? If not, don't worry; you'll soon find out.

In any case, the point is that you learn the order of operations so you know how to send and receive mathematical messages accurately.

Q: What are the steps in the order of operations?



A: Below you'll find the steps in the order of operations. The sooner you memorize these, the easier it'll be for you to simplify expressions.

1st) Work out operations in **parentheses** or other enclosure marks (brackets, absolute value signs, etc.), working from the innermost enclosure marks to the outermost ones.

2nd) Raise terms to **exponents**.

3rd) Perform all **multiplication** and **division** operations, doing whichever operation comes first as you work from left to right.

4th) Use the **neighbor-sign rule**.

5th) **Group** positive numbers with positives and negatives with negatives.

6th) Use the **same-sign rule**.

7th) Use the **mixed-sign rule** to get your answer.

Note: If you don't yet know how to raise numbers to exponents, just look at the first page of the Exponents section (**p. 86**), and you'll find out how.

Q: That's great, but how am I supposed to memorize all these steps? Is there any trick for remembering the order of operations?

A: Yes, there's a memory trick. And if you've read through the section on Positive and Negative Numbers, you already know the last half of the trick. But now you get the whole sentence:

Please **E**at **M**ore **D**essert — **N**ate's **Gr**eat **S**trawberry **M**ousse.

P stands for **P**arentheses.

E stands for **E**xponents.

M stands for **M**ultiplication.

D stands for **D**ivision.

N stands for **N**eighbor-sign rule.

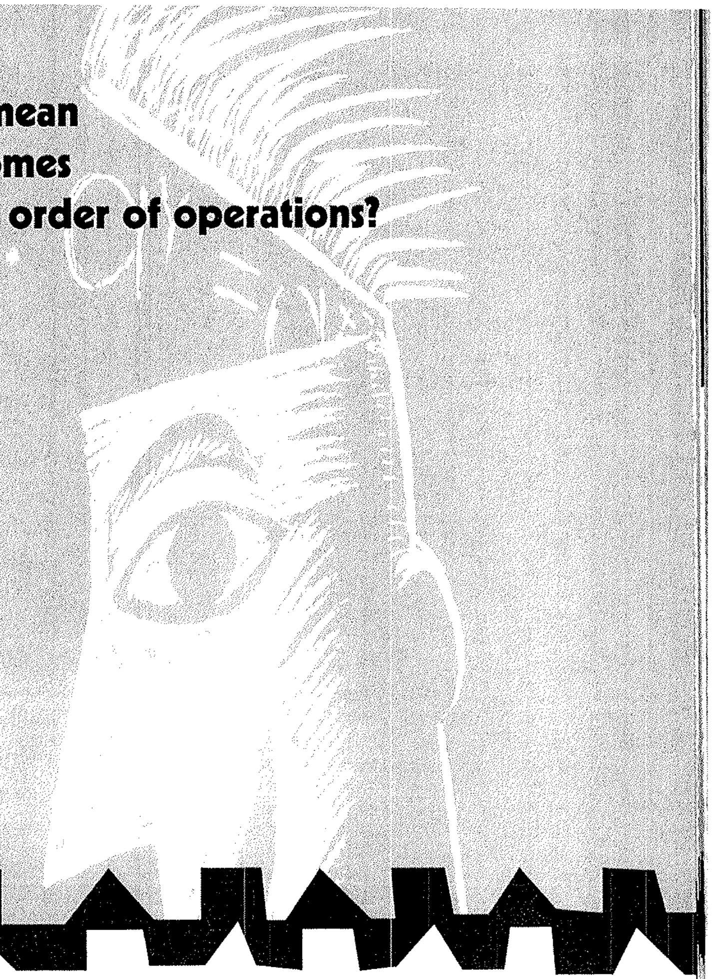
Gr stands for **Gr**ouping.

S stands for **S**ame-sign rule.

M stands for **M**ixed-sign rule.

For those of you who've never had the pleasure, mousse is a light, airy dessert. Some people describe mousse as "pudding with wings."

Q: What exactly does it mean that one operation comes before another in the order of operations? And could I see some examples showing how the order of operations actually works?



A: The fact that one operation is listed before another operation simply means that you do the first operation before the second operation.

For example, the fact that **P** comes before **E** in the list means that you always work out terms inside parentheses before raising terms to their exponents. Here are some examples showing how you use the order of operations in actual problems.

Parentheses before exponents

right:

$$(2+3)^2$$
$$= (5)^2$$
$$= 25$$

wrong:

$$(2+3)^2$$
$$= 2^2 + 3^2$$

Exponents before multiplication

right:

$$4 \cdot 3^2$$
$$= 4 \cdot 9$$
$$= 36$$

wrong:

$$4 \cdot 3^2$$
$$= 12^2$$

Multiplication before mixed-sign rule

right:

$$-3 \cdot 4 + 2 \cdot 8$$
$$= -12 + 16$$
$$= +(16 - 12)$$
$$= +4$$

wrong:

$$-3 \cdot 4 + 2 \cdot 8$$
$$= -3 \cdot (4+2) \cdot 8$$

Tell which operation comes before which other one, then simplify:

a) $(7+1)\cdot(8 \div 2)$

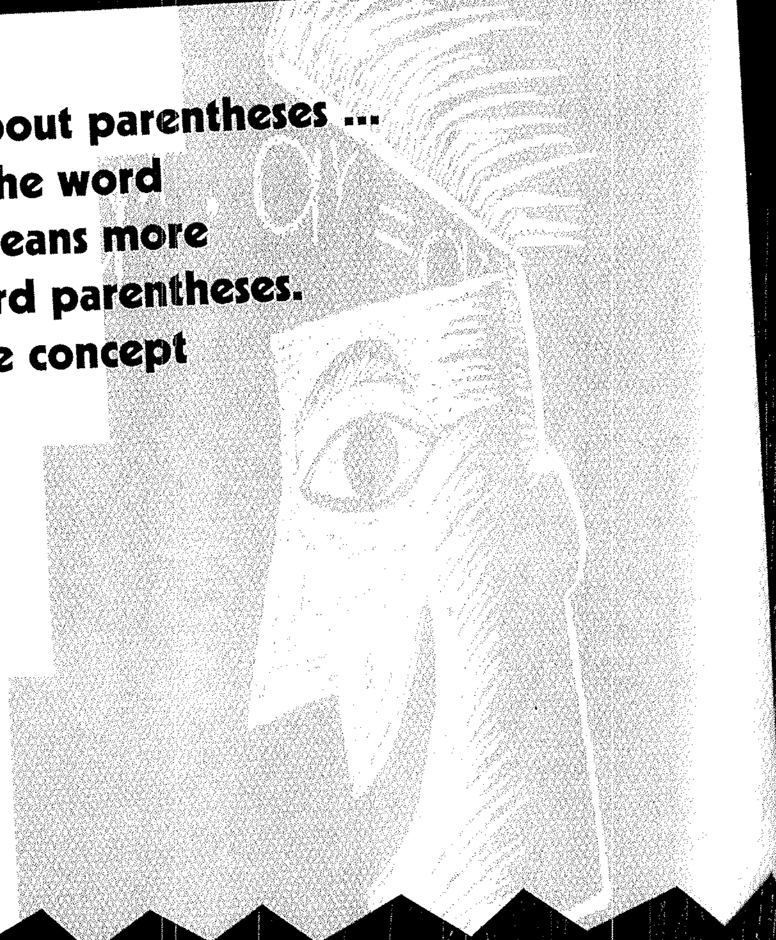
b) $(20-15)^2$

c) $+4^2 - 25$

d) $-15/3 - 11$

e) $-3 + 4 - 6 + 7$

Answers:
a) parentheses before multiplication: 32
b) parentheses before exponents: 25
c) exponents before mixed-sign rule: -9
d) division before same-sign rule: -16
e) grouping before same-sign rule: 2



Q: One question about parentheses ... I've heard that the word "parentheses" means more than just standard parentheses. What exactly is included in the concept of parentheses?

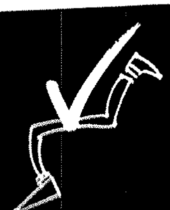
A: **Right.** When you hear "parentheses," remember that it means **all marks that enclose a quantity to set it off from other quantities.** So the term parentheses includes not only normal parentheses: ( ), but also these marks: [ ], { }, and even absolute value marks, | |.

Often you'll see enclosure marks inside enclosure marks, like boxes inside boxes. When you see this, **compute whatever is within the innermost enclosure marks first, then work your way out.** For example, suppose you have this expression:

$$5 \cdot \{2 + 3 \cdot [12 - (16 \div 2)]\}$$

First see which enclosure mark is innermost. Obviously that's parentheses: ( ). So you'd compute the quantity in parentheses first. Then you'd work out the quantity in the next innermost marks, angular brackets: [ ]. Finally you'd work out the quantity inside the remaining enclosure marks, curved brackets: { }.

After working out that quantity, multiply it by the **5** to the left of the curved brackets to get your answer.



Don't simplify. Just tell which enclosure marks you'd work out first, second and third in the following problems:

a) $(18 + [\{12 - 2\} \div 5])$

b) $[14 \div \{8 + (3 \cdot 2)\}]$

c) $\{26 - |(4 \cdot 3) - 8|\}$

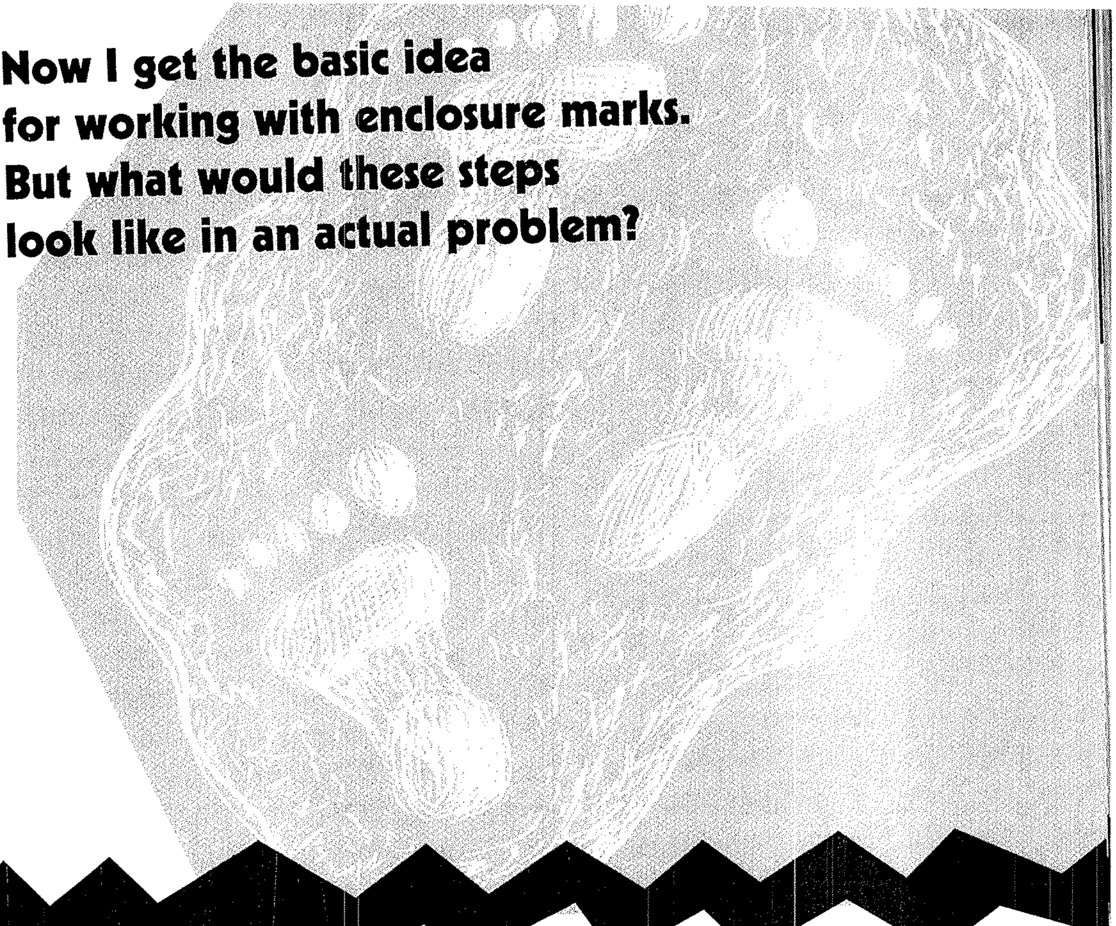
Answers:

a) 1st { }, 2nd [ ], 3rd ( )

b) 1st ( ), 2nd { }, 3rd [ ]

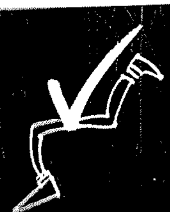
c) 1st ( ), 2nd | |, 3rd { }

Q: Now I get the basic idea for working with enclosure marks. But what would these steps look like in an actual problem?



A: Below you'll see just such an example. Simplify: $4 \cdot \left[18 - \left\{\left(12 \div 2\right) + 4\right\}\right]$

Steps	**Think it out**	**Work it out**
1st) Identify the innermost enclosure marks, compute the value and plug it in.	Innermost is $(12 \div 2)$ $12 \div 2 = 6$	$4 \cdot [18 - \{(12 \div 2) + 4\}]$ $= 4 \cdot [18 - \{6 + 4\}]$
2nd) Identify next innermost enclosure marks, compute the value, plug it in.	Next is $\{6 + 4\}$ $6 + 4 = 10$	$4 \cdot [18 - \{6 + 4\}]$ $= 4 \cdot [18 - 10]$
3rd) Identify final enclosure marks, compute the value, plug it in. Then work out your answer.	Final is $[18 - 10]$ $18 - 10 = 8$	$4 \cdot [18 - 10]$ $= 4 \cdot 8$ $= 32$



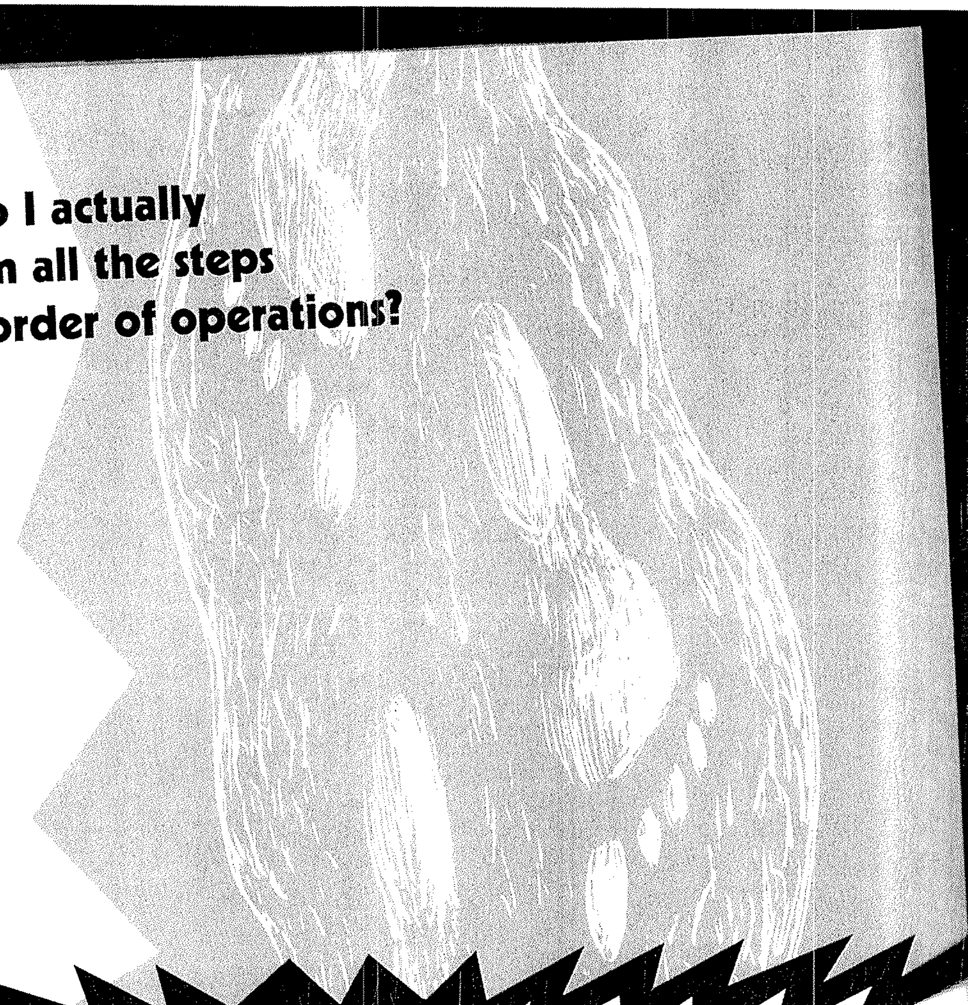
Now you try. Simplify using the inner-to-outer order for enclosure marks:

a) $3 \cdot \{24 - [2 + (5 \cdot 2)]\}$

b) $6[4\{2 + (3 \cdot 1)\} + 2(8 - 3)]$

c) $7[(2 + 3) \cdot (2 - 3)]$

Answers:
a) 36
b) 180
c) -35



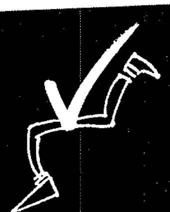
Q: How do I actually perform all the steps in the order of operations?

A: The following problem — **and yes, it's a doozy!** — shows how to use all the steps in the order of operations. Keep in mind that few problems would require you to use **all** these steps.

Simplify using the order of operations: $+2[8-4]-(+3)+5^2-20\div 2$

Steps		Example
Parentheses		$+2[8-4]-(+3)+5^2-20\div 2$
	$=$	$+2[4]-(+3)+5^2-20\div 2$
Exponents	$=$	$+2[4]-(+3)+25-20\div 2$
Multiplication	$=$	$+8-(+3)+25-20\div 2$
Division	$=$	$+8-(+3)+25-10$

Steps		Example	
Neighbor-sign rule	$=$	$+8-3+25-10$	
Grouping	$=$	$+8+25$	$-3-10$
Same-sign rule	$=$	$+(8+25)$	$-(3+10)$
	$=$	$+33$	-13
Mixed-sign rule	$=$	$+(33-13)$	
	$=$	$+20$	



Now try using the order of operations yourself:

a) $8^2-(-10)+25\div 5$

b) $3[4^2-(5+2)]$

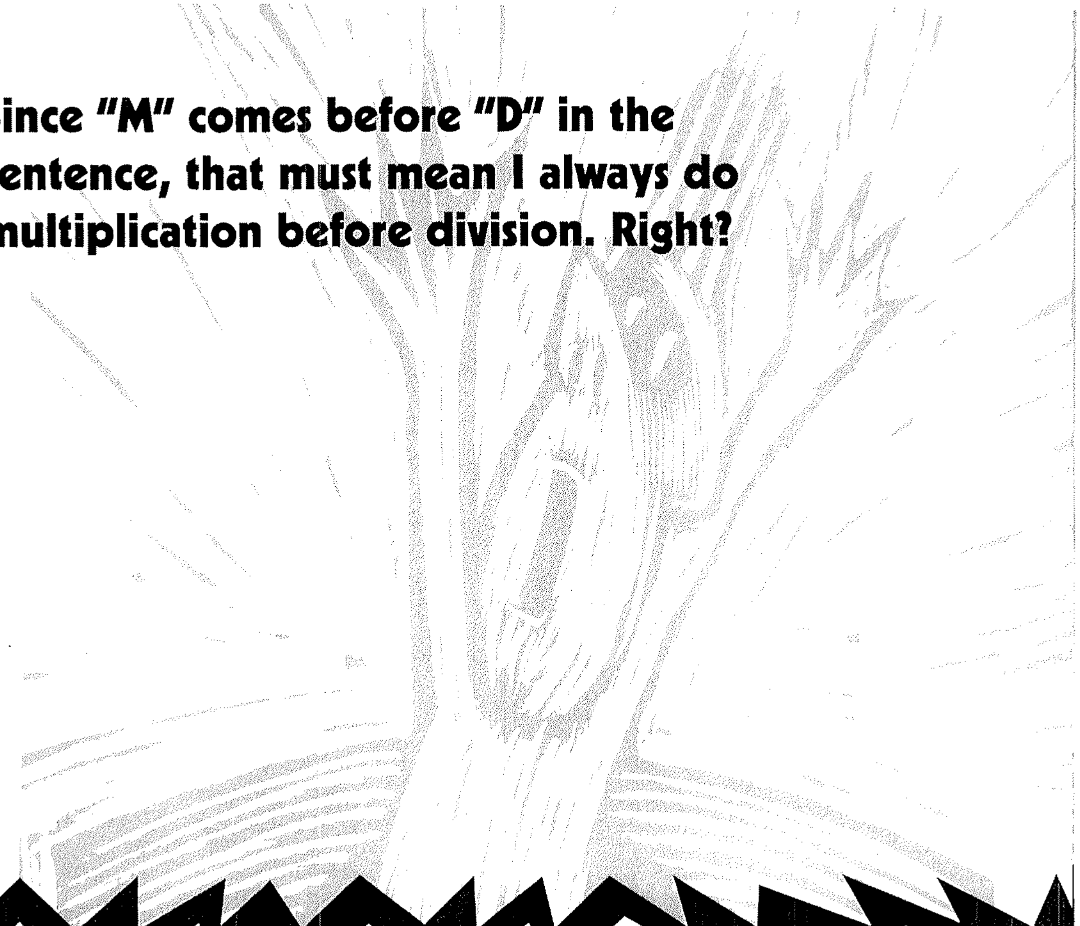
c) $8-(-3)+(-4)-3(10-7)$

d) $(-2)\cdot(-3)+6\{(12\div 6)+2\}$

e) $7+2[(10-8)(5-2)-4]-3^2$

Answers:
a) 79
b) 27
c) -2
d) 30
e) 2

Q: Since "M" comes before "D" in the sentence, that must mean I always do multiplication before division. Right?



A: No, not quite. The rule is that for multiplication and division, you do whichever operation comes first as you read the problem from left to right. If multiplication comes first, you multiply first. But if division comes first, you divide first.

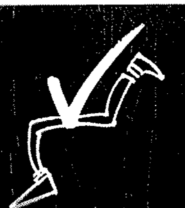
Examples of multiplication and division in the order of operations

$$20 \times 4 \div 2$$
$$= 80 \div 2$$
$$= 40$$

Here you do multiplication before division because the times sign comes first as you read from left to right.

$$20 \div 4 \times 2$$
$$= 5 \times 2$$
$$= 10$$

But here you do division before multiplication because the division sign comes first as you read from left to right.



Try these yourself to see if the order of signs makes a difference:

a) $8 \div 2 \times 4$
b) $2 \times 4 \div 8$
c) $6 \times 5 \div 10$
d) $10 \div 5 \times 6$

Answers:
a) 16
b) 1
c) 3
d) 12

Q: If I see terms like $2 \cdot 3^2$ and $(2 \cdot 3)^2$, I would work these out in exactly the same way. I mean, those silly parentheses don't really affect how I'd work it out, right?

A: Aaaaaagh! No! The "silly" parentheses make all the difference in the world.

Since exponents are worked out before multiplication,

$$2 \cdot 3^2$$
$$= 2 \cdot 9$$
$$= 18$$

Since parentheses are worked out before exponents,

$$(2 \cdot 3)^2$$
$$= 6^2$$
$$= 36$$

Moral of the story: $2 \cdot 3^2 \neq (2 \cdot 3)^2$

<u>Other Examples</u>

$$2x^2 \neq (2x)^2$$

Why?

$$2x^2 = 2x^2$$

But

$$(2x)^2 = (2x) \cdot (2x) = 4x^2$$

$$4a^3 \neq (4a)^3$$

Why?

$$4a^3 = 4a^3$$

But

$$(4a)^3 = (4a) \cdot (4a) \cdot (4a) = 64a^3$$

So $(2x)^2 \neq 2x^2$ and $(4a)^3 \neq 4a^3$.

So remember: when you have a quantity in parentheses raised to an exponent, the parentheses make a huge difference!